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LETTER TO THE EDITOR

A q-deformed oscillator system with quantum group $SL_q(l)$ symmetry

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Abstract. A q-deformed oscillator system with quantum group $SL_q(l)$ symmetry is constructed in terms of the q-deformed boson operators. The energy level degeneracy associated with the irreducible representation of $SL_q(l)$ is analysed, especially for the case that q is a root of unity.

The quantum group [1] symmetries or hidden quantum group symmetries of physical systems have been investigated for many cases, such as the conformal field theories [2, 3], spectra of nuclei [4], the Heisenberg spin model [5], and so on. This letter stresses another important aspect of quantum group symmetry—the relations between the degeneracy of energy levels of the physical system and the irreducible representations of the symmetry quantum group. To this end we first present an l-dimensional q-(deformed) oscillator and then study its energy levels and their degeneracy. The discussion shows that the degeneracy is caused by its quantum group (strictly called quantum algebra) $SL_q(l)$ symmetry. Especially, when q is a root of unity, there appear more degeneracies.

Let us consider a q-oscillator Hamiltonian

$$\hat{H} = \hbar\omega \sum_{j=1}^{l} f_j(N_1, N_2, \ldots, N_l) a_j^+ a_j \tag{1}$$

where

$$\begin{aligned} f_1(N_1, N_2, \ldots, N_l) &= q^{\sum_{i=2}^{l} N_i} \\ f_2(N_1, N_2, \ldots, N_l) &= q^{-N_1 + \sum_{i=3}^{l} N_i} \\ f_3(N_1, N_2, \ldots, N_l) &= q^{-(N_1+N_2) + \sum_{i=4}^{l} N_i} \\ &\ldots \\ f_k(N_1, N_2, \ldots, N_l) &= q^{-\sum_{i=1}^{k-1} N_i + \sum_{i=k+1}^{l} N_i} \\ f_l(N_1, N_2, \ldots, N_l) &= q^{-\sum_{i=1}^{l-1} N_i}. \end{aligned} \tag{2}$$

The l classes $\{a_i^- = a_i, a_i^+, N_i\}(i = 1, 2, \ldots, l)$ of q-deformed boson operators [6–8] satisfy the basic relations

$$\begin{aligned} &a_i a_j^+ - q^{-\delta_{ij}} a_j^+ a_i = \delta_{ij} q^{N_i} \qquad [N_i, N_j] = 0 \\ &[N_j, a_i^\pm] = \pm\delta_{ij} a_i^\pm \qquad [a_i^\pm, a_j^\pm] = 0. \end{aligned} \tag{3}$$

When $q \to 1$, $\hat{H}$ becomes the Hamiltonian of the usual n-dimensional oscillator. We need to point out that the $l=2$ case of the above q-oscillator has been built by Kulish [9] with

$$\hat{H} = q^{N}2a_1^+a_1 + q^{-N_1}a_2^+a_2. \tag{4}$$

On the q-deformed Fock space [7]

$$\mathscr{F}_q(l)\text{: span}\{|n_1, n_2, \ldots, n_l\rangle = a_1^{+n_1}a_2^{+n_2}\ldots a_l^{+n_l}|0\rangle | a_k|0\rangle = N_k|0\rangle = 0$$
$$n_k \in \mathbb{Z}^+ = \{0, 1, 2, \ldots\}, k = 1, 2, \ldots\}$$

the Hamiltonian $\hat{H}$ is diagonal, i.e.

$$\hat{H}|n_1, n_2, \ldots, n_l\rangle = E(n_1, n_2, \ldots, n_l)|n_1, n_2, \ldots, n_l\rangle \tag{5}$$

where the energy levels are

$$E(n_1, n_2 \ldots + n_l) = \hbar\omega[n_1 + n_2 + \ldots + n_l]. \tag{6}$$

These eigenvalues are quite different from that for the non-coupling q-oscillator with the Hamiltonian

$$\hat{H}' = \hbar\omega \sum_{i=1}^{l} a_i^+ a_i \tag{7}$$

and the corresponding eigenvalues

$$\mathscr{E}(n_1, n_2, \ldots, n_l) = \hbar\omega([n_1] + [n_2] + \ldots + [n_l]). \tag{8}$$

In fact, for the q-oscillator defined by (1) and (2), as long as $n_1 + n_2 + \ldots + n_l$ is fixed, the energy levels $E(n_1, n_2, \ldots, n_l)$ are the same for different $(n_1, n_2, \ldots, n_l)$, that is to say, the energy levels $\{E(n_1, n_2, \ldots, n_l)|n_1 + n_2 + \ldots + n_l = N\} \in \mathbb{Z}^+$ (the fixed N belongs to $\mathbb{Z}^+$) are degenerate. The degree of the degeneracy is

$$D(n_1, n_2, \ldots, n_l) = \frac{(l+N-1)!}{(l-1)!\,N!}. \tag{9}$$

Since the appearance of degeneracy in the usual quantum mechanics implies the existence of a symmetry group, we naturally ask: what is the symmetry 'group' for the degenerate q-oscillator defined by (1)?

In order to answer this question, we notice that the important equation

$$a_i^+ a_i = [N_i] \qquad i = 1, 2, \ldots, l \tag{10}$$

holds on the q-Fock space $\mathscr{F}_q(l)$. Then

$$\hat{H} = \hbar\omega[N_1 + N_2 + \ldots + N_l] \tag{11}$$

where $[f] = (q^f - q^{-f})/(q - q^{-1})$. From (10) we easily observe that $[N_1 + N_2 + \ldots + N_l]$ commutes with each of the following operators

$$E_i^+ = a_i^+ a_{i+1} \qquad E_i^- = F_i = a_{i+1}^+ a_i \qquad H_i = N_i - N_{i+1}$$
$$i = 1, 2, \ldots, l-1 \tag{12}$$

on the q-Fock spaces. Fortunately, equations (12) are just the q-deformed boson realization of the quantum group $SL_q(l) = U_q(A_{l-1})$ first given in [7].

As we pointed out in [7], the subspace $V^{[N]} = \text{span}\{|n_1, n_2, \ldots, n_l\rangle | n_1 + n_2 + \ldots + n_l = N\}$ is invariant under the action of $SL_q(l)$. And it carries an irreducible representation of $SL_q(l)$ when q is not a root of unity. This invariant subspace is just the eigenspace of eigenvalue $E=\hbar\omega[N]$; that is to say, all the vectors in $V^{[N]}$ are degenerate. Therefore, the above discussion shows that the $SL_q(l)$ symmetry of the Hamiltonian (1) causes the degeneracy of degree $D(n_1, n_2, \ldots, n_l)$ for the $l-D$ q-oscillator, where $D(n_1, n_2, \ldots, n_l)$ just equals the dimension of irreducible representation of $SL_q(l)$.

Now, we turn to discuss the non-generic case that $q(\neq 1)$ is a root of unity, or $q^p=\pm 1 (p=2,3,4,\ldots)$. In this case, $[kp]=0$ $(k\in\mathbb{Z}^+)$ and

$$\begin{aligned} &E(n_1+k_1p, n_2+k_2p, \ldots, n+kp)=E(n_1, n_2, \ldots, n) \\ &k_i\in\mathbb{Z}^+ \qquad i=1,2,\ldots,l. \end{aligned} \tag{13}$$

This fact tells us that more degenerate energy levels exist when q is a root of unity. What is the origin of the new degeneracy?

In fact, when $q^p=1$, we can prove from (3) that $(a_i^{\pm})^{kp}$ $(k\in\mathbb{Z}^+)$ commute with all the q-boson operators and

$$(a_i^+)^{kp}(a_i)^{k'p}=\begin{cases} (a_i^+)^{(k-k')p}\left\{\prod_{m=1}^{p}[N+p-m]\right\}^{k'} & \text{if } k\geq k' \\ \left\{\prod_{m=1}^{p}[N+p-m]\right\}^{k}(a_i)^{(k'-k)p} & \text{if } k<k' \end{cases} \tag{14}$$

since

$$\begin{aligned} &[a_i^{\pm kp}, H]=0 \\ &[q^{\pm N_i}, H]=0 \end{aligned} \qquad k\in\mathbb{Z}^+ \qquad i=1,2,\ldots. \tag{15}$$

The Abelian algebra $\mathscr{A}$ generated by $(a_i^{\pm})^{kp}$ and $q^{\pm N_i}$ is the symmetry algebra of the q-oscillator system defined by (1). Because $\mathscr{A}$ is infinite dimensional, the degree of degeneracy caused by $\mathscr{A}$ is infinite. Thus, when q is a root of unity, the symmetry algebra for the q-oscillator (1) is $SL_q(l)\otimes\mathscr{A}$.

In the terminology of the representation theory for $SL_q(l)$, when the new degeneracy described by (14) appears, the irreducible representation on subspace $V^{[N]}$ invariant for $q^p\neq 1$ is no longer irreducible [10–12]. This is because the $[kp]=0$ causes new invariant subspaces in $V^{[N]}$. Finally, we point out that the extra degeneracy caused by $q^p=1$ is completely quantum and has no classical correspondence.

Finally, it should be pointed out that a formalism of the spectrum-generating quantum group of oscillators is given by Macfarlane and Majid, but is not directly related to the present letter. In their work, the fermionic group generators for osp(1, 2), or their quantum group equivalent, is introduced [13].

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